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Higher-derivative 3-algebras

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Abstract: Starting with the $\mathcal{N} = 8$ supersymmetric Yang-Mills theory on D2-branes and incorporating higher-derivative corrections to lowest nontrivial order, we perform a duality to derive the Lorentzian 3-algebra theory along with a set of derivative corrections. We find that these corrections can be expressed entirely in terms of intrinsic 3-algebra quantities: the 3-bracket and covariant derivatives. Our analysis is performed for both bosonic and fermionic terms. We conjecture that the derivative corrections we obtain are relevant for Euclidean 3-algebra theories as well.

Keywords: Duality in Gauge Field Theories, D-branes, M-Theory.

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Contents

1. Introduction

It is known that at low energy the world volume modes of N M2-branes decouple from the eleven-dimensional gravity in the bulk leading to an $\mathcal{N} = 8$ superconformal field theory in three dimensions. This superconformal theory has an SO(8) R-symmetry which can be identified with the geometric SO(8) symmetry acting on the eight transverse directions of the M2-branes. Although we have understood this theory through its symmetries, it was not clear for over a decade how to write a model describing three dimensional $\mathcal{N} = 8$ superconformal field theory.

In a series of paper Bagger and Lambert [1 − 3] and also Gustavsson [4] have constructed an action which is consistent with all the symmetries of a 3D $\mathcal{N} = 8$ superconformal field theory; namely it is conformal invariant with 16 supercharges and has an SO(8) R-symmetry acting on eight scalar fields. Therefore this model has the potential to describe the world-volume theory of multiple M2-branes.

This construction relies on the introduction of an algebraic structure called a "Lie 3-algebra" characterized by 4-index structure constants, $f^{ABC}{}_{D}$ and a bi-linear metric h^{AB}. The structure constants satisfy a fundamental identity which is essentially a generalization of the Jacobi identity of the Lie 2-algebra. Depending on whether the metric is positive definite or indefinite we distinguish two cases: Euclidean and Lorentzian theories.[1] Although the Euclidean theory, originally proposed by Bagger and Lambert, can only describe a theory with SO(4) gauge symmetry where $f^{ABCD} = \epsilon^{ABCD}$, the Lorentzian theory may be written for any classical Lie algebra [6 − 8].

[1] See [5] for an alternative treatment.

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Even though in the original Lorentzian theories there were potential ghost-like degrees of freedom, a variant has been proposed that has been argued to be unitary and describe multiple M2-branes [9, 10]. The argument is as follows. One modifies the theory by gauging a shift symmetry for one of the "null" coordinates X^I_+ by introducing a gauge field. The other null coordinate X^I_- is frozen as a result of the equation of motion of the gauge field. Therefore the resultant theory is manifestly ghost-free. Indeed, using the Higgs mechanism of ref. [11] it was shown [8, 12] that the theory reduces to maximally supersymmetric Yang-Mills in three dimensions whose gauge coupling is the vev of the scalar field. This result indicates that the ghost free Lorentzian theory is closely related to SYM theory. However in [13] it was shown that starting from maximally supersymmetric 3D Yang-Mills theory and using a duality transformation due to de Wit, Nicolai and Samtleben [15 − 17], one can directly obtain the ghost-free Lorentzian 3-algebra theory.[2] Since it can be derived from SYM, the final theory is manifestly equivalent to it on-shell. Though it does have enhanced R-symmetry as well as superconformal symmetry off-shell, it is the D2-brane theory on-shell for any *finite* vev of the gauge-singlet scalar field.

On the other hand at higher orders in α' the world-volume theory of multiple D2-branes is believed to be described by some non-Abelian generalization of the DBI action. Therefore, one would expect that the 3-algebra theories just represent the lowest order of the full effective action describing the world-volume of multiple M2-branes. Therefore it should be interesting to study non-linear corrections to 3-algebra theories. One straightforward approach is to consider these corrections in the context of Lorentzian 3-algebras, where as indicated above they should be *derivable* from the SYM theory.

Accordingly, in this article we extend the considerations of [13] when higher-derivative corrections are taken into account. More precisely starting with the $\mathcal{N} = 8$ supersymmetric Yang-Mills theory on D2-branes and incorporating higher-derivative corrections to lowest nontrivial order, we perform a duality to derive the Lorentzian 3-algebra theory along with a set of derivative corrections given by non-Abelian F^4 terms [18]. We will show that these corrections assemble themselves neatly into the basic objects of a 3-algebra, namely the 3-bracket and covariant derivatives. This holds for both bosonic and fermionic terms and we provide explicit forms for the leading correction in both cases.

Finally we conjecture that the derivative corrections we have obtained here, being independent of the details of the 3-algebra, should be relevant for Euclidean 3-algebra theories as well. This conjecture in principle enlarges the potential applicability of the results in this paper to a wider class of 3-algebras beyond the Lorentzian-signature ones. However, because the 3-bracket for us is totally antisymmetric, our results can be immediately generalized at this stage only to maximally supersymmetric ($\mathcal{N} = 8$) Euclidean 3-algebras, of which the sole example is the Bagger-Lambert A_4 theory [3]. It may be possible in the future to extend these considerations to 3-algebra theories with lower supersymmetry such as those discussed in refs. [19, 20] (see also [21]).

The rest of the paper is organized as follows. In section two we will set our notation

[2]The same mechanism was subsequently used to derive globally $N = 8$ supersymmetric actions from supergravity [14].

by reviewing the construction of ref. [13]. In section three we will extend the results to incorporate bosonic non-Abelian F^4 terms and the corresponding scalar terms. In section four we discuss some general features of these higher order corrections. In section five we obtain the SO(8) covariant fermionic terms to the same order in α'. Finally we present a conjecture and our conclusions.

2. Review

We would like to consider the maximally supersymmetric interacting super Yang-Mills Lagrangian in 2+1 dimensions based on an arbitrary Lie algebra $\mathcal{G}$ whose bosonic action in leading order is given by:

$$\mathcal{L} = \mathrm{Tr}\left(-\frac{1}{4g_{\mathrm{YM}}^2}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}D_\mu X^i D^\mu X^i - \frac{g_{\mathrm{YM}}^2}{4}[X^i,X^j][X^j,X^i]\right)\ , \tag{2.1}$$

Here A_μ is a gauge connection on $\mathcal{G}$. The field strength and the covariant derivatives are defined as:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu] \qquad \text{and} \qquad D_\mu = \partial_\mu - [A_\mu, \cdot\,]\ . \tag{2.2}$$

The X^is are seven matrix valued scalar fields transforming as vectors under the SO(7) R-symmetry group.

In [13] it was shown that this Lagrangian can be brought to the form of the Lorentzian Bagger-Lambert or 3-algebra field theory proposed in [6–8], or more precisely to the "gauged" version of the above theory described in [9, 10]. Here we first review the results of [13].

We proceed by introducing two new fields B_μ and ϕ that are adjoints of $\mathcal{G}$. In terms of these new fields the dNS duality transformation [15–17] is the replacement:

$$\mathrm{Tr}\left(-\frac{1}{4g_{\mathrm{YM}}^2}F^{\mu\nu}F_{\mu\nu}\right) \to \mathrm{Tr}\left(\frac{1}{2}\epsilon^{\mu\nu\lambda}B_\mu F_{\nu\lambda} - \frac{1}{2}\left(D_\mu\phi - g_{\mathrm{YM}}B_\mu\right)^2\right)\ . \tag{2.3}$$

We see that in addition to the gauge symmetry $\mathcal{G}$, the new action has a noncompact Abelian gauge symmetry that we can call $\tilde{\mathcal{G}}$, which has the same dimension as the original gauge group $\mathcal{G}$. This symmetry consists of the transformations:

$$\delta\phi = g_{\mathrm{YM}}M\ , \qquad \delta B_\mu = D_\mu M\ , \tag{2.4}$$

where $M(x)$ is an arbitrary matrix, valued in the adjoint of $\mathcal{G}$. Clearly B_μ is the gauge field for the shift symmetries $\tilde{\mathcal{G}}$. Note that both in eq. (2.3) and eq. (2.4), the covariant derivative D_μ is the one defined in eq. (2.2).

If one chooses the gauge $D^\mu B_\mu = 0$ to fix the shift symmetry, the degree of freedom of the original Yang-Mills gauge field A_μ can be considered to reside in the scalar ϕ. In this sense one can think of ϕ as morally the dual of the original A_μ [15–17]. Alternatively we can choose the gauge $\phi = 0$, in which case the same degree of freedom resides in B_μ. The equivalence of the r.h.s. to the l.h.s. of eq. (2.3) can be conveniently seen by going to the

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latter gauge. Once $\phi = 0$ then B_μ is just an auxiliary field and one can integrate it out to find the usual YM kinetic term for $F_{\mu\nu}$.

We can now proceed to study the dNS-duality transformed of the bosonic sector of $\mathcal{N} = 8$ Yang-Mills theory. Its Lagrangian is:

$$\begin{aligned}\mathcal{L} =& \mathrm{Tr}\left(\frac{1}{2}\epsilon^{\mu\nu\lambda}B_\mu F_{\nu\lambda} - \frac{1}{2}\left(D_\mu\phi - g_{\mathrm{YM}}B_\mu\right)^2 - \frac{1}{2}D_\mu X^i D^\mu X^i\right.\\ &\left. -\frac{g_{\mathrm{YM}}^2}{4}[X^i, X^j][X^j, X^i]\right) \; .\end{aligned} \tag{2.5}$$

The gauge-invariant kinetic terms for the eight scalar fields have a potential SO(8) invariance, which can be exhibited as follows. First rename $\phi(x) \to X^8(x)$. Then the scalar kinetic terms become $-\frac{1}{2}\hat{D}_\mu X^I \hat{D}^\mu X^I$, where:

$$\begin{aligned}\hat{D}_\mu X^i &= D_\mu X^i = \partial_\mu X^i - [A_\mu, X^i], \quad i = 1, 2, \ldots, 7\\ \hat{D}_\mu X^8 &= D_\mu X^8 - g_{\mathrm{YM}}B_\mu = \partial_\mu X^8 - [A_\mu, X^8] - g_{\mathrm{YM}}B_\mu \; .\end{aligned} \tag{2.6}$$

Defining the constant 8-vector:

$$g_{\mathrm{YM}}^I = (0, \ldots, 0, g_{\mathrm{YM}}) \; , \quad I = 1, 2, \ldots, 8 \; , \tag{2.7}$$

the covariant derivatives can together be written:

$$\hat{D}_\mu X^I = D_\mu X^I - g_{\mathrm{YM}}^I B_\mu \; . \tag{2.8}$$

One can now uniquely write the SYM action in a form that is SO(8)-invariant under transformations that rotate both the fields X^I and the coupling-constant vector g_{YM}^I:

$$\begin{aligned}\mathcal{L} =& \mathrm{Tr}\Big(\frac{1}{2}\epsilon^{\mu\nu\lambda}B_\mu F_{\nu\lambda} - \frac{1}{2}\hat{D}_\mu X^I \hat{D}^\mu X^I\\ &- \frac{1}{12}\left(g_{\mathrm{YM}}^I[X^J, X^K] + g_{\mathrm{YM}}^J[X^K, X^I] + g_{\mathrm{YM}}^K[X^I, X^J]\right)^2\Big) \; .\end{aligned} \tag{2.9}$$

The final step is to replace g_{YM}^I by a scalar field X_+^I that is constrained to be a constant.[3] This proceeds as described in [13] and we will describe it again in the following section where we address higher-derivative terms. The fermionic contributions also must be added, and these too will be described in what follows.

3. F^4 terms

The aim of this section is to redo the procedure of the previous section for subleading terms of the three dimensional theory. The subleading terms consist of F^4 with four derivative interactions of the scalar fields. To find the explicit terms we note that the leading order

[3] Flux quantization in the original theory implies the matrix-valued scalars have a periodicity $X^I \sim X^I + X_+^I \mathbb{I}$. We thank Juan Maldacena for emphasizing this to us.

terms in the action can be found from reduction of the ten dimensional pure gauge Yang-Mills theory. Therefore to get the higher derivative terms for the three dimensional theory we will start from ten dimensional F^4 terms given by [18][4]

$$L^{(10)} = \frac{1}{12}\mathrm{Tr}\Big[F_{MN}F_{RS}F^{MR}F^{NS} + \frac{1}{2}F_{MN}F^{NR}F_{RS}F^{SM} - \frac{1}{4}F_{MN}F^{MN}F_{RS}F^{RS} - \frac{1}{8}F_{MN}F_{RS}F^{MN}F^{RS}\Big], \tag{3.1}$$

where $M, N, R, S = 0, \cdots, 9$. The aim is now to reduce this action to three dimensions. To do that we decompose the indices to $\mu, \nu, \rho, \sigma = 0, 1, 2$ and $i, j, k, l = 1, \cdots, 7$. Then the Yang-Mills plus F^4 terms lead to the following Lagrangian:

$$L^{(4)} = L^{(2)} + \sum_{i=1}^{6} \mathrm{Tr}\ L_i^{(4)}, \tag{3.2}$$

where

$$L^{(2)} = -\frac{1}{4g_{\mathrm{YM}}^2}F_{\mu\nu}F^{\mu\nu} \tag{3.3}$$

$$L_1^{(4)} = \frac{1}{12g_{\mathrm{YM}}^4}\Big[F_{\mu\nu}F_{\rho\sigma}F^{\mu\rho}F^{\nu\sigma}+\frac{1}{2}F_{\mu\nu}F^{\nu\rho}F_{\rho\sigma}F^{\sigma\mu}-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} - \frac{1}{8}F_{\mu\nu}F_{\rho\sigma}F^{\mu\nu}F^{\rho\sigma}\Big]$$

$$\begin{aligned} L_2^{(4)} = \frac{1}{12g_{\mathrm{YM}}^2}\Big[&F_{\mu\nu}\ D^\mu X^i\ F^{\rho\nu}\ D_\rho X^i + F_{\mu\nu}\ D_\rho X^i\ F^{\mu\rho}\ D^\nu X^i - 2F_{\mu\rho}\ F^{\rho\nu}\ D^\mu X^i\ D_\nu X^i \\ &-2F_{\mu\rho}\ F^{\rho\nu}\ D_\nu X^i\ D^\mu X^i - F_{\mu\nu}\ F^{\mu\nu}\ D^\rho X^i\ D_\rho X^i - \frac{1}{2}F_{\mu\nu}\ D_\rho X_i\ F_{\mu\nu}\ D_\rho X_i\Big] \\ &-\frac{1}{12}\left(\frac{1}{2}F_{\mu\nu}\ F^{\mu\nu}\ X^{ij}\ X^{ij} + \frac{1}{4}F_{\mu\nu}\ X^{ij}\ F^{\mu\nu}\ X^{ij}\right) \end{aligned} \tag{3.4}$$

$$L_3^{(4)} = -\frac{1}{6}\Big(D^\mu X^i\ D^\nu X^j F_{\mu\nu} + D^\nu X^j\ F_{\mu\nu}\ D^\mu X^i + F_{\mu\nu}\ D^\mu X^i\ D^\nu X^j\Big)\ X^{ij} \tag{3.5}$$

$$\begin{aligned} L_4^{(4)} = \frac{1}{12}\Big[&D_\mu X^i\ D_\nu X^j\ D^\nu X^i\ D^\mu X^j + D_\mu X^i\ D_\nu X^j\ D^\mu X^j\ D^\nu X^i \\ &+D_\mu X^i\ D_\nu X^i\ D^\nu X^j\ D^\mu X^j - D_\mu X^i\ D^\mu X^i\ D_\nu X^j\ D^\nu X^j \\ &-\frac{1}{2}D_\mu X^i\ D_\nu X^j\ D^\mu X^i\ D^\nu X^j\Big] \end{aligned} \tag{3.6}$$

$$\begin{aligned} L_5^{(4)} = \frac{g_{\mathrm{YM}}^2}{12}\Big[&X^{kj}\ D_\mu X^k\ X^{ij}\ D^\mu X^i + X^{ij}\ D_\mu X^k\ X^{ik}\ D^\mu X^j \\ &-2X^{kj}\ X^{ik}\ D_\mu X^j\ D^\mu X^i - 2X^{ki}\ X^{jk}\ D_\mu X^j\ D^\mu X^i \\ &-X^{ij}\ X^{ij}\ D_\mu X^k\ D^\mu X^k - \frac{1}{2}X^{ij}\ D_\mu X^k\ X^{ij}\ D^\mu X^k\Big] \end{aligned} \tag{3.7}$$

$$L_6^{(4)} = \frac{g_{\mathrm{YM}}^4}{12}\Big[X^{ij}X^{kl}X^{ik}X^{jl}+\frac{1}{2}X^{ij}X^{jk}X^{kl}X^{li}-\frac{1}{4}X^{ij}X^{ij}X^{kl}X^{kl}-\frac{1}{8}X^{ij}X^{kl}X^{ij}X^{kl}\Big] \tag{3.8}$$

[4] We are using units in which $\alpha' = \frac{1}{2\pi}$.

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Following the previous section the aim is to rewrite the above Lagrangian in terms of the new fields, B_μ, X^8 such that the obtained Lagrangian will be manifestly SO(8) invariant. It is useful to proceed in two steps. First we simply rewrite the Lagrangian in terms of the Poincare dual field strength defined by:

$$\tilde{F}_\mu \equiv \frac{1}{2}\epsilon_{\mu\nu\lambda}F^{\nu\lambda} \tag{3.9}$$

Note that in our conventions (with a $(-++)$ metric), the inverse transformation is $F_{\mu\nu} = -\epsilon_{\mu\nu\lambda}\tilde{F}^\lambda$. Later we will replace $\tilde{F}$ by an independent field B_μ that will be subjected to constraints via the equations of motion, leading back to the original action.

Replacing $F_{\mu\nu}$ in terms of $\tilde{F}_\mu$ everywhere in the preceding Lagrangian, we end up with:

$$\begin{aligned}
&L^{(2)} + L_1^{(4)} + L_2^{(4)} + L_3^{(4)} = \\
&\quad \mathrm{Tr}\Bigg[\frac{1}{2g_{\rm YM}^2}\tilde{F}_\mu\tilde{F}^\mu + \frac{1}{12g_{\rm YM}^4}\left(\tilde{F}_\mu\tilde{F}^\mu\tilde{F}_\nu\tilde{F}^\nu + \frac{1}{2}\tilde{F}_\mu\tilde{F}_\nu\tilde{F}^\mu\tilde{F}^\nu\right) \\
&\qquad +\frac{1}{12g_{\rm YM}^2}\Big(2\tilde{F}^\mu\tilde{F}_\nu\, D^\nu X^i\, D_\mu X^i - 2\tilde{F}^\mu\tilde{F}_\mu\, D_\nu X^i\, D^\nu X^i + 2\tilde{F}^\mu\tilde{F}^\nu\, D_\mu X^i\, D_\nu X^i \\
&\qquad +\tilde{F}^\mu\, D^\nu X^i\, \tilde{F}_\nu\, D_\mu X^i - \tilde{F}^\mu\, D^\nu X^i\, \tilde{F}_\mu\, D_\nu X^i + \tilde{F}^\mu\, D_\mu X^i\, \tilde{F}^\nu\, D_\nu X^i\Big) \\
&\qquad +\frac{1}{12}\left(\tilde{F}^\mu\, \tilde{F}_\mu\, X^{ij}\, X^{ij} + \frac{1}{2}\tilde{F}^\mu\, X^{ij}\, \tilde{F}_\mu\, X^{ij}\right) \\
&\qquad +\frac{1}{6}\epsilon_{\rho\mu\nu}\left(\tilde{F}^\rho\, D^\mu X^i\, D^\nu X^j + D^\nu X^j\, \tilde{F}^\rho\, D^\mu X^i + D^\mu X^i\, D^\nu X^j\, \tilde{F}^\rho\right)X^{ij}\Bigg]
\end{aligned} \tag{3.10}$$

Here we have written only the terms involving $\tilde{F}$, as the remaining ones $L_4^{(4)}, L_5^{(4)}, L_6^{(4)}$ are obviously unaffected by our substitution.

Let us now perform a dNS duality, as in the previous section, but in the presence of the above higher-derivative corrections. Introducing again an independent 1-form (matrix-valued) field B_μ, it is easy to see that the above action can be replaced with one where $\tilde{F}$ appears only in the Chern-Simons interaction $\tilde{F}_\mu B^\mu$:

$$\begin{aligned}
L^{(2)} + L_1^{(4)} + L_2^{(4)} + L_3^{(4)} = \mathrm{Tr}\Bigg[&\tilde{F}_\mu B^\mu - \frac{g_{\rm YM}^2}{2}B_\mu B^\mu \\
&+\frac{g_{\rm YM}^4}{12}\left(B_\mu B^\mu B_\nu B^\nu + \frac{1}{2}B_\mu B_\nu B^\mu B^\nu\right) \\
&+\frac{g_{\rm YM}^2}{12}\Big(2B^\mu B_\nu\, D^\nu X^i\, D_\mu X^i - 2B^\mu B_\mu\, D_\nu X^i\, D^\nu X^i + 2B^\mu B^\nu\, D_\mu X^i\, D_\nu X^i \\
&+B^\mu\, D^\nu X^i\, B_\nu\, D_\mu X^i - B^\mu\, D^\nu X^i\, B_\mu\, D_\nu X^i + B^\mu\, D_\mu X^i\, B^\nu\, D_\nu X^i\Big) \\
&+\frac{g_{\rm YM}^4}{12}\left(B^\mu\, B_\mu\, X^{ij}\, X^{ij} + \frac{1}{2}B^\mu\, X^{ij}\, B_\mu\, X^{ij}\right) \\
&+\frac{g_{\rm YM}^2}{6}\epsilon_{\rho\mu\nu}\left(B^\rho\, D^\mu X^i\, D^\nu X^j + D^\nu X^j\, B^\rho\, D^\mu X^i + D^\mu X^i\, D^\nu X^j\, B^\rho\right)X^{ij}\Bigg]
\end{aligned} \tag{3.11}$$

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To show that this substitution is correct, simply integrate out the field B order by order (truncating at quartic order, since that is all the input we had to start with) using its equation of motion. It is easy to check that this brings the above Lagrangian to the form:

$$L^{(2)} + \mathrm{Tr}(L_1^{(4)} + L_2^{(4)} + L_3^{(4)}) + \mathcal{O}(F^6). \tag{3.12}$$

We now use this form, depending on the new field B_μ, to rewrite the Lagrangian in an SO(8) invariant way. For this, introduce the field X^8 and replace B_μ, everywhere it occurs, by $-\frac{1}{g_{\mathrm{YM}}}(D_\mu X^8 - g_{\mathrm{YM}} B_\mu)$. There is now a shift symmetry as in eq. (2.4) using which one can set $X^8 = 0$ and we get back to the above action. The utility of this transformation will be that in more general gauges, X^8 can carry the dynamical degree of freedom.

As explained in eqs. (2.6), (2.7), (2.8), it is useful to write the coupling constant formally as an 8-vector, since this allows us to express all the covariant derivatives in a unified manner as $\hat{D}_\mu X^I, I = 1, 2, \cdots, 8$. Then under the above replacement, eq. (3.10) becomes:[5]

$$\begin{aligned}
\mathrm{Tr}\Bigg[&\frac{1}{2}\epsilon^{\mu\nu\rho} B_\mu F_{\nu\rho} - \frac{1}{2}\hat{D}_\mu X^8 \hat{D}^\mu X^8 \\
&+\frac{1}{12}\Big(\hat{D}_\mu X^8 \hat{D}^\mu X^8 \hat{D}_\nu X^8 \hat{D}^\nu X^8 + \frac{1}{2}\hat{D}_\mu X^8 \hat{D}_\nu X^8 \hat{D}^\mu X^8 \hat{D}^\nu X^8\Big) \\
&+\frac{1}{12}\Big(2\hat{D}^\mu X^8\, \hat{D}_\nu X^8\, \hat{D}^\nu X^i\, \hat{D}_\mu X^i - 2\hat{D}^\mu X^8\, \hat{D}_\mu X^8\, \hat{D}_\nu X^i\, \hat{D}^\nu X^i \\
&\qquad +2\hat{D}^\mu X^8\, \hat{D}^\nu X^8\, \hat{D}_\mu X^i\, \hat{D}_\nu X^i + \hat{D}^\mu X^8\, \hat{D}^\nu X^i\, \hat{D}_\nu X^8\, \hat{D}_\mu X^i \\
&\qquad -\hat{D}^\mu X^8\, \hat{D}^\nu X^i\, \hat{D}_\mu X^8\, \hat{D}_\nu X^i + \hat{D}^\mu X^8\, \hat{D}_\mu X^i\, \hat{D}^\nu X^8\, \hat{D}_\nu X^i\Big) \\
&\qquad +\frac{g_{\mathrm{YM}}^2}{12}\Big(\hat{D}^\mu X^8\, \hat{D}_\mu X^8\, X^{ij}\, X^{ij} + \frac{1}{2}\hat{D}^\mu X^8\, X^{ij}\, \hat{D}_\mu X^8\, X^{ij}\Big) \\
&+\frac{g_{\mathrm{YM}}}{6}\epsilon_{\rho\mu\nu}\Big(\hat{D}^\rho X^8\, \hat{D}^\mu X^i\, \hat{D}^\nu X^j + \hat{D}^\nu X^j\, \hat{D}^\rho X^8\, \hat{D}^\mu X^i + \hat{D}^\mu X^i\, \hat{D}^\nu X^j\, \hat{D}^\rho X^8\Big) X^{ij}\Bigg]
\end{aligned} \tag{3.13}$$

It is now straightforward, though a little messy, to see that the leading order terms given in equation (2.1) plus $\sum_{i=1}^{6} \mathrm{Tr}\ L_i^{(4)}$ can be written in the SO(8) invariant terms as follows:

$$\begin{aligned}
\mathrm{Tr}\Bigg[&\frac{1}{2}\epsilon^{\mu\nu\rho} B_\mu F_{\nu\rho} - \frac{1}{2}\hat{D}_\mu X^I \hat{D}^\mu X^I \\
&+\frac{1}{12}\Big(\hat{D}_\mu X^I\, \hat{D}_\nu X^J\, \hat{D}^\nu X^I\, \hat{D}^\mu X^J + \hat{D}_\mu X^I\, \hat{D}_\nu X^J\, \hat{D}^\mu X^J\, \hat{D}^\nu X^I \\
&\qquad +\hat{D}_\mu X^I\, \hat{D}_\nu X^I\, \hat{D}^\nu X^J\, \hat{D}^\mu X^J - \hat{D}_\mu X^I\, \hat{D}^\mu X^I\, \hat{D}_\nu X^J\, \hat{D}^\nu X^J \\
&\qquad -\frac{1}{2}\hat{D}_\mu X^I\, \hat{D}_\nu X^J\, \hat{D}^\mu X^I\, \hat{D}^\nu X^J\Big)
\end{aligned} \tag{3.14}$$

[5] Using integration by parts and cyclicity of the trace one can show that the $\tilde{F}^\mu D_\mu X^8$ term vanishes.

$$
\begin{aligned}
&+\frac{1}{12}\Big(\frac{1}{2}X^{LKJ}\,\hat{D}_\mu X^K\,X^{LIJ}\,\hat{D}^\mu X^I+\frac{1}{2}X^{LIJ}\,\hat{D}_\mu X^K\,X^{LIK}\,\hat{D}^\mu X^J\\
&\qquad -X^{LKJ}\,X^{LIK}\,\hat{D}_\mu X^J\,\hat{D}^\mu X^I-X^{LKI}\,X^{LJK}\,\hat{D}_\mu X^J\,\hat{D}^\mu X^I\\
&\qquad -\frac{1}{3}X^{LIJ}\,X^{LIJ}\,\hat{D}_\mu X^K\,\hat{D}^\mu X^K-\frac{1}{6}X^{LIJ}\,\hat{D}_\mu X^K\,X^{LIJ}\,\hat{D}^\mu X^K\Big)\\
&\qquad\qquad\qquad -\frac{1}{6}\epsilon_{\rho\mu\nu}\hat{D}^\rho X^I\,\hat{D}^\mu X^J\,\hat{D}^\nu X^K X^{IJK}-V(X)\Big]
\end{aligned}
$$

where

$$
X^{IJK}=g^I_{\rm YM}[X^J,X^K]+g^J_{\rm YM}[X^K,X^I]+g^K_{\rm YM}[X^I,X^J] \tag{3.15}
$$

Here $V(X)$ is the potential:

$$
\begin{aligned}
V(X)=\;&\frac{1}{12}X^{IJK}X^{IJK}+\frac{1}{9\times 12}\Big[X^{NIJ}X^{NKL}X^{MIK}X^{MJL}\\
&+\frac{1}{2}X^{NIJ}X^{MJK}X^{NKL}X^{MLI}-\frac{1}{4}X^{NIJ}X^{NIJ}X^{MKL}X^{MKL}\\
&-\frac{1}{8}X^{NIJ}X^{MKL}X^{NIJ}X^{MKL}\Big]
\end{aligned} \tag{3.16}
$$

Once we have SO(8) covariance, we are free to replace the fixed vector of coupling constants $g^I_{\rm YM}$ by any arbitrary vector with the same modulus. The last step is to replace these constants by a set of scalar fields X^I_+ and introduce another scalar X^I_- as well as a gauge field $C^{\mu,I}$ with the kinetic term:

$$
(C^{\mu\,I}-\partial^\mu X^I_-)\partial^\mu X^I_+ \tag{3.17}
$$

As explained in refs. [10, 13], this has the effect of constraining the vector X^I_+ to be an arbitrary constant which we can then identify with $g^I_{\rm YM}$.

Thus the final form of our derivative-corrected action is as in eqs. (3.14) and (3.16), with the covariant derivatives replaced by:

$$
\hat{D}_\mu X^I=\partial_\mu-[A_\mu,X^I]-B_\mu X^I_+ \tag{3.18}
$$

and the commutator terms eq. (3.15) replaced by the Lorentzian 3-algebra triple product:

$$
X^{IJK}=X^I_+[X^J,X^K]+X^J_+[X^K,X^I]+X^K_+[X^I,X^J] \tag{3.19}
$$

This must be supplemented, of course, with fermionic terms as well as gauge-fixing terms for the various local symmetries. We will discuss the fermions in detail in a subsequent section.

To summarize, in this section we have used dNS duality to re-write the three dimensional $\mathcal{N}=8$ supersymmetric Yang-Mills theory, including the first nontrivial derivative corrections, in a form which is manifestly SO(8) invariant. We now turn to a discussion of the generality of this result.

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4. Generality of the result and higher order terms

Encouraged by what we have found, we would like in this section to ask how general the result is. Is it true that to any order, the derivative correction computed for $\mathcal{N} = 8$ SYM in 3d can be re-expressed in SO(8) invariant form? Specifically we wish to understand whether achieving SO(8) invariance depends on the specific combination of F^4 terms appearing in eq. (3.1). If this is not the case, in other words if enhanced SO(8) is generically present for any higher order F^n terms that one can think of writing down in 10d, then it would not be such a miracle. But in fact, as we will see below, SO(8) enhancement does *not* hold for generic higher-order corrections. The specific combination occurring in eq. (3.1), which arises from string theory, is essential for the result that we found in order F^4, and a similar situation is expected to hold in higher orders.

Instead of considering the most general case, we will find it illuminating to start with a simplified approach. Consider an Abelian SYM theory in 10d. Let us now postulate a generic quartic correction to the 10d Lagrangian, namely:

$$L^{(4)}_{10d} = \lambda_1\, F_{AB}F^{AB}F_{CD}F^{CD} + \lambda_2\, F^{A}_{\ B}F^{B}_{\ C}F^{C}_{\ D}F^{D}_{\ A} \tag{4.1}$$

where we have put arbitrary coefficients in front of the two possible quartic terms. (In this section we set $g_{\rm YM} = 1$ for notational simplicity.) After reducing to 3d, the field strength terms can be dualized to 1-forms as before (using $\tilde{F}_\mu = \frac{1}{2}\epsilon_{\mu\nu\lambda}F^{\nu\lambda}$) and we find:

$$L^{(4)}_{\rm gauge} = (4\lambda_1 + 2\lambda_2)\, \tilde{F}_\mu\tilde{F}^\mu\tilde{F}_\nu\tilde{F}^\nu \tag{4.2}$$

Note that two different tensor structures in 10d have reduced to the same one in 3d. This is because of the duality between 1-forms and 2-forms in 3d. On the other hand, the terms involving ∂X are found to be:

$$\begin{aligned} L^{(4)}_{\partial X} =& -(8\lambda_1 + 4\lambda_2)\, \partial_\mu X^i\partial^\mu X^i\, \tilde{F}_\nu\tilde{F}^\nu + 4\lambda_2\, \partial_\mu X^i\partial_\nu X^i\, \tilde{F}^\mu\tilde{F}^\nu \\ &+ 4\lambda_1\, \partial_\mu X^i\partial^\mu X^i\partial_\nu X^j\partial^\nu X^j + 2\lambda_2\, \partial_\mu X^i\partial_\nu X^i\partial^\mu X^j\partial^\nu X^j \end{aligned} \tag{4.3}$$

where as usual the indices $i, j = 1, 2, \cdots, 7$. For the Abelian case eqs. (4.2), (4.3) make up the whole reduced action to this order, since commutator terms are absent.

Now let us ask if the above expression has SO(8) invariance after performing dNS duality. To quartic order this duality merely replaces $\tilde{F}_\mu$ everywhere in the quartic terms by B_μ (as we will see, this is not the the case from order 6 onwards). After that, we replace B_μ by $-\partial_\mu X^8$. The result for the quartic action $L^{(4)}_{3d} = L^{(4)}_{\rm gauge} + L^{(4)}_{\partial X}$ is:

$$\begin{aligned} L^{(4)}_{3d} =& \ (4\lambda_1 + 2\lambda_2)\, \partial_\mu X^8\partial^\mu X^8\partial_\nu X^8\partial^\nu X^8 - (8\lambda_1 + 4\lambda_2)\, \partial_\mu X^i\partial^\mu X^i\partial_\nu X^8\partial^\nu X^8 \\ &+4\lambda_2\, \partial_\mu X^i\partial_\nu X^i\partial^\mu X^8\partial^\nu X^8 + 4\lambda_1\, \partial_\mu X^i\partial^\mu X^i\partial_\nu X^j\partial^\nu X^j \\ &+2\lambda_2\, \partial_\mu X^i\partial_\nu X^i\partial^\mu X^j\partial^\nu X^j \end{aligned} \tag{4.4}$$

Now it is easy to see that the above action is equal to the SO(8) invariant combination:

$$4\lambda_1\, \partial_\mu X^I\partial^\mu X^I\partial_\nu X^J\partial^\nu X^J + 2\lambda_2\, \partial_\mu X^I\partial_\nu X^I\partial^\mu X^J\partial^\nu X^J \tag{4.5}$$

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where $I, J = 1, 2, \cdots, 8$, but only if the following constraint is satisfied:

$$\lambda_2 = -4\lambda_1 \tag{4.6}$$

Without this constraint, $L_{3d}^{(4)}$ cannot be recast in SO(8) invariant form.

In light of this simple computation, we may go back to the previous section and see if that computation, specialized to the Abelian case, satisfies our constraint above. Once we treat all F's as commuting, we find that the four coefficients in eq. (3.1) collapse to two independent coefficients corresponding to $\lambda_1 = -\frac{1}{32}$ and $\lambda_2 = \frac{1}{8}$. Therefore the above constraint is satisfied. This explains why we found SO(8) invariance in the previous section and makes it clear that this was crucially dependent on using the corrections that arise in string theory (which evidently "knows" about this constraint) and would not have worked for generic correction terms.

In fact, for the Abelian case it is an old result [22, 23] that SO(8) invariance can be obtained for the full DBI action by performing a duality. We summarize that argument here after translating it into our conventions for ease of comparison, and presenting in the more "modern" dNS form which admits a non-Abelian generalization. Start with the $(2+1)$d DBI action:

$$L = -\sqrt{-\det\left(g_{\mu\nu} + \frac{1}{g_{\rm YM}} F_{\mu\nu}\right)} \tag{4.7}$$

This action is equivalent to the following action involving a new independent field B_μ:

$$L = \frac{1}{2}\epsilon^{\mu\nu\lambda} B_\mu F_{\nu\lambda} - \sqrt{-\det(g_{\mu\nu} + g_{\rm YM}^2 B_\mu B_\nu)} \tag{4.8}$$

To prove equivalence, simply integrate out B_μ from the latter action and recover the former action.

Now noting that in static gauge, $g_{\mu\nu} = \eta_{\mu\nu} + \partial_\mu X^i \partial_\nu X^i$, and making the replacement:

$$B_\mu \to -\frac{1}{g_{\rm YM}} \hat{D}_\mu X^8 = -\frac{1}{g_{\rm YM}} \left(\partial_\mu X^8 - B_\mu X_+^8\right) \tag{4.9}$$

we find that the action eq. (4.8) turns into:

$$L = \frac{1}{2}\epsilon^{\mu\nu\lambda} B_\mu F_{\nu\lambda} - \sqrt{-\det(\eta_{\mu\nu} + \hat{D}_\mu X^I \hat{D}_\nu X^I)} \tag{4.10}$$

Hence SO(8) invariance is achieved. It is easily seen that this subsumes the special (quartic, Abelian) case that we discussed at the beginning of this section.

The considerations in this section support our conjecture that the entire non-Abelian D2-brane action can be recast in SO(8) invariant form, and constitute an important (though long-known) consistency check on it, since if it works for the non-Abelian case then it must necessarily work for the Abelian reduction. But to prove the (non-Abelian) conjecture in general is more difficult, essentially because the full non-Abelian D-brane action is not yet known. Having treated the bosonic terms to lowest nontrivial order in α', we next turn to treatment of the fermionic terms.

5. Fermionic terms

The fermionic terms of the action can also be obtained from 10 dimensional supersymmetric gauge theory reduced to three dimensions. To do this we first need the supersymmetrized DBI action at α'^2 level. Then we may reduce the fermionic terms to three dimension in the same way as we have done for the bosonic part in a previous section. The aim would be to rewrite the resulting fermionic terms in SO(8) invariant form.

Let us start with the Abelian case, which has essentially been treated in the older literature. We will provide a re-derivation which stresses more explicitly the promotion to SO(8) invariance. This will be a guide in studying the non-Abelian case. Start with the following DBI Lagrangian in 10 dimensions [24]:

$$L = -\sqrt{-\det(\eta_{MN} + F_{MN} - 2\bar{\lambda}\Gamma_M\partial_N\lambda + \bar{\lambda}\Gamma^P\partial_M\lambda\,\bar{\lambda}\Gamma_P\partial_N\bar{\lambda})} \tag{5.1}$$

Upon dimensional reduction to 3 dimensions, this reduces to:

$$-\left(-\begin{vmatrix} \eta_{\mu\nu} + F_{\mu\nu} - 2\bar{\lambda}\Gamma_\mu\partial_\nu\lambda + \bar{\lambda}\Gamma^\rho\partial_\mu\lambda\,\bar{\lambda}\Gamma_\rho\partial_\nu\lambda + \bar{\lambda}\Gamma^i\partial_\mu\lambda\,\bar{\lambda}\Gamma^i\partial_\nu\lambda & -\partial_\mu X^i \\ \partial_\nu X^i - 2\bar{\lambda}\Gamma_i\partial_\nu\lambda & \eta_{ij} \end{vmatrix}\right)^{\frac{1}{2}} \tag{5.2}$$

which can be rewritten as:

$$\begin{aligned} -\Big[-\det\Big(&\eta_{\mu\nu} + \partial_\mu X^i\partial_\nu X^i - 2\partial_\mu X^i\bar{\lambda}\Gamma^i\partial_\nu\lambda + \bar{\lambda}\Gamma^i\partial_\mu\lambda\bar{\lambda}\Gamma^i\partial_\nu\lambda + F_{\mu\nu} \\ &-2\bar{\lambda}\Gamma_\mu\partial_\nu\lambda + \bar{\lambda}\Gamma^\rho\partial_\mu\lambda\,\bar{\lambda}\Gamma_\rho\partial_\nu\bar{\lambda}\Big)\Big]^{\frac{1}{2}} \end{aligned} \tag{5.3}$$

This can now be re-expressed as:

$$-\sqrt{-\det\left(\tilde{G}_{\mu\nu} + D_\mu X^i D_\nu X^i + \mathcal{F}_{\mu\nu}\right)} \tag{5.4}$$

where:

$$\begin{aligned} \tilde{G}_{\mu\nu} &= \eta_{\mu\nu} - 2\bar{\lambda}\Gamma_{(\mu}\partial_{\nu)}\lambda + \bar{\lambda}\Gamma^\rho\partial_\mu\lambda\,\bar{\lambda}\Gamma_\rho\partial_\nu\lambda \\ \mathcal{F}_{\mu\nu} &= F_{\mu\nu} - 2\bar{\lambda}\Gamma_{[\mu}\partial_{\nu]}\lambda - 2\partial_{[\mu}X^i\bar{\lambda}\Gamma^i\partial_{\nu]}\lambda \\ \hat{D}_\mu X^i &= \partial_\mu X^i - \bar{\lambda}\Gamma^i\partial_\mu\lambda \end{aligned} \tag{5.5}$$

Now following the result in ref. [24], the above action is dual to:

$$\frac{1}{2}\epsilon^{\mu\nu\rho}\Big(B_\mu - \frac{1}{g_{\rm YM}}\partial_\mu X^8\Big)\mathcal{F}_{\nu\rho} - \sqrt{-\det(\tilde{G}_{\mu\nu} + \hat{D}_\mu X^I\hat{D}_\nu X^I)} \tag{5.6}$$

where

$$\hat{D}_\mu X^8 \equiv \partial_\mu X^8 - g_{\rm YM}B_\mu \tag{5.7}$$

and $\hat{D}_\mu X^i = D_\mu X^i,\ i = 1, \cdots, 7$ which was defined above.

This expression does not look SO(8) invariant, both for the Chern-Simons term and the covariant derivative, but we can argue that in fact both these are SO(8) invariant. First consider the covariant derivatives. For the $g_{\rm YM}B_\mu$ term we proceed as was explained for the bosonic case. However, the fermionic term which appears in $\hat{D}_\mu X^i$ is absent in $\hat{D}_\mu X^8$. This seems to pose a problem for SO(8) invariance. In fact, the quantity:

$$\Pi^i_\mu = \partial X^i_\mu - \bar{\lambda}\Gamma^i\partial_\mu\lambda \tag{5.8}$$

is a supercovariant quantity which occurs in many formulae. So the question is to understand why

$$\Pi^8_\mu = \partial_\mu X^8 - \bar{\lambda}\Gamma^8\partial_\mu\lambda \tag{5.9}$$

does not appear. This would be required to form the SO(8) vector Π^I_μ

As explained in ref. [25], because we are in static gauge both with respect to coordinate transformations and supersymmetries, the fermion λ is really a 16-component fermion descending from the 32-component fermion θ in the covariant D-brane formalism. Starting with the original fermionic variable θ we define:

$$\theta_1 = \frac{1}{2}(1+\Gamma^8)\theta, \qquad \theta_2 = \frac{1}{2}(1-\Gamma^8)\theta \tag{5.10}$$

(what we call Γ^8 is referred to as Γ^{11} in ref. [25]). Then static gauge is chosen by putting $\theta_2 = 0$, and rename θ_1 as λ. Hence:

$$\Gamma^8\lambda = \lambda \tag{5.11}$$

It follows that:

$$\bar{\lambda}\Gamma^8\partial_\mu\lambda = \bar{\lambda}\partial_\mu\lambda = \frac{1}{2}\partial_\mu(\bar{\lambda}\lambda) \tag{5.12}$$

(using the identity $\bar{\lambda}\chi = \bar{\chi}\lambda$ for Majorana-Weyl spinors in 10d). Therefore:

$$\Pi^8_\mu = \partial_\mu\left(X^8 - \frac{1}{2}(\bar{\lambda}\lambda)\right) \tag{5.13}$$

and the second term can be removed by a shift of X^8. This explains why the covariant derivatives are in fact SO(8) covariant.

For the Chern-Simons term something similar happens. The extra term compared to the bosonic case is proportional to:

$$\epsilon^{\mu\nu\rho}\partial_\mu X^8\left(\bar{\lambda}\Gamma_\nu\partial_\rho\lambda + \partial_\nu X^i\bar{\lambda}\Gamma^i\partial_\rho\lambda\right) \tag{5.14}$$

Consider the first term in the above expression. To make it covariant we would like to write it as:

$$\epsilon^{\mu\nu\rho}\partial_\mu X^8\,\bar{\lambda}\Gamma_\nu\partial_\rho\lambda = \epsilon^{\mu\nu\rho}\partial_\mu X^8\,\bar{\lambda}\Gamma_\nu\Gamma^8\partial_\rho\lambda \to \epsilon^{\mu\nu\rho}\partial_\mu X^I\,\bar{\lambda}\Gamma_\nu\Gamma^I\partial_\rho\lambda \tag{5.15}$$

where the first step is an identity (because $\Gamma^8\lambda = \lambda$) and in the second step we have added a piece:

$$\epsilon^{\mu\nu\rho}\partial_\mu X^i\,\bar{\lambda}\Gamma_\nu\Gamma^i\partial_\rho\lambda \tag{5.16}$$

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As we now show, this extra piece is equal to zero, which justifies adding it to make the above term SO(8) covariant. We have:

$$\begin{aligned}\epsilon^{\mu\nu\rho}\partial_\mu X^i\,\bar\lambda\Gamma_\nu\Gamma^i\partial_\rho\lambda &= \frac{1}{2}\epsilon^{\mu\nu\rho}\partial_\mu X^i\,\bar\lambda(\Gamma_\nu\Gamma^i-\Gamma^i\Gamma_\nu)\partial_\rho\lambda\\ &= \frac{1}{4}\epsilon^{\mu\nu\rho}\partial_\mu X^i\,\partial_\rho\Big(\bar\lambda(\Gamma_\nu\Gamma^i-\Gamma^i\Gamma_\nu)\lambda\Big)\end{aligned}\tag{5.17}$$

which is zero on integration by parts. (Here we have used the identity $\bar\lambda\Gamma^{MN}\chi=\bar\chi\Gamma^{MN}\lambda$).)

Things work similarly for the second term in eq. (5.14):

$$\partial_\mu X^8\partial_\nu X^i\,\bar\lambda\Gamma^i\partial_\rho\lambda=\partial_\mu X^8\partial_\nu X^i\,\bar\lambda\Gamma^i\Gamma^8\partial_\rho\lambda\tag{5.18}$$

To make this covariant we need to add:

$$\frac{1}{2}\partial_\mu X^i\partial_\nu X^j\,\bar\lambda\Gamma^{ij}\partial_\rho\lambda=\frac{1}{4}\partial_\mu X^i\partial_\nu X^j\,\partial_\rho\Big(\bar\lambda\Gamma^{ij}\lambda\Big)\tag{5.19}$$

but this is again zero on partial integration. Thus we have shown that the Abelian fermionic Chern-Simons terms can be written in SO(8) invariant form as:

$$\epsilon^{\mu\nu\rho}\partial_\mu X^I\left(\bar\lambda\Gamma_\nu\Gamma^I\partial_\rho\lambda+\frac{1}{2}\partial_\nu X^J\,\bar\lambda\Gamma^{IJ}\partial_\rho\lambda\right)\tag{5.20}$$

Turning now to the non-Abelian case of interest to us, the relevant fermionic terms at α'^2 level in ten dimensional supersymmetric gauge theory are given by [26, 27][6]

$$\begin{aligned}L_{\rm fer}=\mathrm{Str}\bigg(&\frac{i}{2}\bar\lambda\Gamma^M D_M\lambda+\frac{i}{4}\bar\lambda\Gamma_M D^N\lambda F^{MR}F_{RN}-\frac{i}{8}\bar\lambda\Gamma_{MNR}D_S\lambda F^{MN}F^{RS}\\ &-\frac{1}{16}\bar\lambda\Gamma^M D^N\lambda\,\bar\lambda\Gamma_N D_M\lambda\bigg).\end{aligned}\tag{5.21}$$

We proceed as follows. First reduce the action to 3 dimensions and then try to rewrite the obtained action in an SO(8) invariant form. Of course one also needs to take the symmetrized trace *Str*. We note however that the first term is easy to deal with. In fact, dimensionally reducing to three dimensions one gets

$$\frac{i}{2}\mathrm{Tr}\bigg(\bar\lambda\Gamma^\mu D_\mu\lambda+g_{\rm YM}\bar\lambda\Gamma^i[X^i,\lambda]\bigg),\tag{5.22}$$

which can be written as follows:

$$\frac{i}{2}\mathrm{Tr}\bigg(\bar\lambda\Gamma^\mu D_\mu\lambda+\frac{1}{2}\bar\lambda\Gamma^{IJ}[X^I,X^J,\lambda]\bigg),\tag{5.23}$$

where

$$[X^I,X^J,\lambda]=g^I_{\rm YM}[X^J,\lambda]-g^J_{\rm YM}[X^I,\lambda].\tag{5.24}$$

[6]Here we have not considered terms like $F\bar\lambda\Gamma\lambda\bar\lambda\Gamma\lambda$ which from the string theory point of view are of order of α'^2g^3 while the terms we are considering are of order of α'^2g^2. For details see [26, 27].

The last term in eq. (5.21) can also be reduced to three dimensions, leading to

$$-\frac{1}{16}\mathrm{Str}\Big(\bar\lambda\Gamma^\mu D^\nu\lambda\,\bar\lambda\Gamma_\mu D_\nu\lambda+g_{\mathrm{YM}}\bar\lambda\Gamma^i D^\nu\lambda\,\bar\lambda\Gamma_\nu[X^i,\lambda]+g_{\mathrm{YM}}\bar\lambda\Gamma^\mu[X^i,\lambda]\,\bar\lambda\Gamma^i D_\mu\lambda$$
$$+g_{\mathrm{YM}}^2\bar\lambda\Gamma^i[X^j,\lambda]\,\bar\lambda\Gamma^j[X^i,\lambda]\Big) \quad (5.25)$$

Using our notation the above action can be recast in the following SO(8) invariant form

$$-\frac{1}{16}\mathrm{Str}\Big(\bar\lambda\Gamma^\mu D^\nu\lambda\,\bar\lambda\Gamma_\mu D_\nu\lambda+\frac{1}{4}g_{\mathrm{YM}}^2\bar\lambda\Gamma^{IJ}[X^K,X^L,\lambda]\,\bar\lambda\Gamma^{KL}[X^I,X^J,\lambda]$$
$$+\frac{1}{2}\bar\lambda\Gamma^\mu[X^I,X^J,\lambda]\,\bar\lambda\Gamma^{IJ}D_\mu\lambda+\frac{1}{2}\bar\lambda\Gamma^{IJ}D^\nu\lambda\,\bar\lambda\Gamma_\nu[X^I,X^J,\lambda]\Big) \quad (5.26)$$

Of course we still need to take the symmetrized trace Str.

The second and third terms in eq. (5.21) are more involved. For these terms it is useful to first expand the Str (of course at the end we will again rewrite the action in terms of Str). Doing so, we get

$$\begin{aligned}&\mathrm{Str}\Big(\frac{i}{4}\bar\lambda\Gamma_M D^N\lambda F^{MR}F_{RN}-\frac{i}{8}\bar\lambda\Gamma_{MNR}D_S\lambda F^{MN}F^{RS}\Big)\\&\quad=\frac{1}{3!}\mathrm{Tr}\Big[\Big(\frac{i}{4}\bar\lambda\Gamma_M D^N\lambda F^{MR}F_{RN}-\frac{i}{8}\bar\lambda\Gamma_{MNR}D_S\lambda F^{MN}F^{RS}\Big)\\&\qquad\qquad\Big(\frac{i}{4}\bar\lambda\Gamma_M D^N\lambda F_{RN}F^{MR}-\frac{i}{8}\bar\lambda\Gamma_{MNR}D_S\lambda F^{RS}F^{MN}\Big)\\&\qquad\qquad+4\text{ more pairs obtained from permutations of }F_{MN}\text{ and }\lambda\Big].\end{aligned} \quad (5.27)$$

We note, however, that to reduce and convert the obtained action to the SO(8) invariant terms we do not need the four extra pairs coming from the permutations. As soon as we get the SO(8) invariant from of the first two pairs, the others can be obtained by an obvious permutation of λ's and $\hat{D}X^J$'s. So in what follows we just concentrate on the first two pairs.

Reducing the above part of the fermionic action from the first two pairs, one finds:

$$\begin{aligned}\frac{1}{3!}\mathrm{Tr}\Big[\Big(&\frac{i}{4}\Big\{\frac{1}{g_{\mathrm{YM}}^2}\bar\lambda\Gamma_\mu D^\nu\lambda\,F^{\mu\rho}F_{\rho\nu}-\bar\lambda\Gamma_\mu D^\nu\lambda\,D^\mu X^l D_\nu X^l-\frac{1}{g_{\mathrm{YM}}}\bar\lambda\Gamma^i D^\nu\lambda\,D^\rho X^i F_{\rho\nu}\\&\quad-g_{\mathrm{YM}}\bar\lambda\Gamma^i D^\nu\lambda\,X^{il}D_\nu X^l+\bar\lambda\Gamma_\mu[X^j,\lambda]\,F^{\mu\rho}D_\rho X^j\\&\quad-g_{\mathrm{YM}}\bar\lambda\Gamma^i[X^j,\lambda]\,D^\rho X^i D_\rho X^j+g_{\mathrm{YM}}^2\bar\lambda\Gamma_\mu[X^j,\lambda]D^\mu X^l X^{lj}\\&\quad+g_{\mathrm{YM}}^3\bar\lambda\Gamma^i[X^j,\lambda]X^{il}X^{lj}\Big\}\\&-\frac{i}{8}\Big\{\frac{1}{g_{\mathrm{YM}}^2}\bar\lambda\Gamma_{\mu\nu\rho}D_\sigma\lambda\,F^{\mu\nu}F^{\rho\sigma}+\bar\lambda\Gamma_{\mu\nu\rho}[X^k,\lambda]\,F^{\mu\nu}D^\rho X^k-\frac{1}{g_{\mathrm{YM}}}\bar\lambda\Gamma_{\mu\nu l}D_\sigma\lambda\,F^{\mu\nu}D^\sigma X^l\\&\quad+g_{\mathrm{YM}}\bar\lambda\Gamma_{\mu\nu l}[X^k,\lambda]\,F^{\mu\nu}X^{lk}+\frac{2}{g_{\mathrm{YM}}}\bar\lambda\Gamma_{\mu j\rho}D_\sigma\lambda\,D^\mu X^j F^{\rho\sigma}\\&\quad+2g_{\mathrm{YM}}\bar\lambda\Gamma_{\mu j\rho}[X^k,\lambda]\,D^\mu X^j D^\rho X^k-2\bar\lambda\Gamma_{\mu jl}D_\sigma\lambda\,D^\mu X^j D^\sigma X^l\end{aligned}$$

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$$+ 2g_{\rm YM}^2\bar\lambda\Gamma_{\mu jl}[X^k,\lambda]\; D^\mu X^j X^{lk} + \bar\lambda\Gamma_{ij\rho}D_\sigma\lambda\; X^{ij}F^{\rho\sigma}$$
$$+ g_{\rm YM}^2\bar\lambda\Gamma_{ij\rho}[X^k,\lambda]\; X^{ij}D^\rho X^k - g_{\rm YM}\bar\lambda\Gamma_{ijl}D_\sigma\lambda\; X^{ij}D^\sigma X^l$$
$$+ g_{\rm YM}^3\bar\lambda\Gamma_{ijl}[X^k,\lambda]\; X^{ij}X^{lk}\Big\}\Big) + \Big(\text{the same terms with } F\leftrightarrow F\Big) + \cdots\Big]. \qquad (5.28)$$

Now the task is to rewrite these terms in SO(8) invariant form. To do this, following the procedure of the previous section we should first dualize F to B field and then use the shift symmetry to replace B by $\hat{D}X^8$. Using the properties of 3D gamma matrices and dropping terms which are zero on shell[7] one arrives at

$$\begin{aligned}
\frac{i}{8}{\rm Str}\Big[&2\bar\lambda\Gamma_\mu\Gamma^{IJ}D_\nu\lambda\hat{D}^\mu X^I\hat{D}^\nu X^J - 2\bar\lambda\Gamma_\mu D^\nu\lambda\hat{D}^\mu X^I\hat{D}_\nu X^I \qquad (5.29)\\
&+\bar\lambda\Gamma^{IJKL}D_\nu\lambda\; X^{IJK}\hat{D}^\nu X^L - \bar\lambda\Gamma^{IJ}D_\nu\lambda\; X^{IJK}\hat{D}^\nu X^K\\
&-2\bar\lambda\Gamma^{IJ}[X^J,X^K,\lambda]\hat{D}^\mu X^I\hat{D}_\mu X^K - 2\bar\lambda[X^I,X^J,\lambda]\hat{D}^\mu X^I\hat{D}_\mu X^J\\
&-2\bar\lambda\Gamma^{\mu\nu}[X^I,X^J,\lambda]\hat{D}_\mu X^I\hat{D}_\nu X^J - 2\bar\lambda\Gamma_{\mu\nu}\Gamma^{IJ}[X^J,X^K,\lambda]\hat{D}^\mu X^I\hat{D}^\nu X^K\\
&+\bar\lambda\Gamma_\mu\Gamma^{IJ}[X^K,X^L,\lambda]\hat{D}^\mu X^I X^{JKL} - \bar\lambda\Gamma_\mu[X^I,X^J,\lambda]\hat{D}^\mu X^K X^{IJK}\\
&-\frac{1}{3}\bar\lambda\Gamma_\mu\Gamma^{IJKL}[X^L,X^M,\lambda]X^{IJK}\hat{D}^\mu X^M - \bar\lambda\Gamma_\mu\Gamma^{IJ}[X^K,X^L,\lambda]X^{IJK}\hat{D}^\mu X^L\\
&-\frac{1}{6}\bar\lambda\Gamma^{IJKL}[X^M,X^N,\lambda]X^{IJL}X^{KMN} - \frac{1}{2}\bar\lambda\Gamma^{IJ}[X^K,X^L,\lambda]X^{IJM}X^{KLM}\Big].
\end{aligned}$$

To summarize this section, we have found the SO(8) invariant fermionic terms to lowest nontrivial order in α' and they are contained in the sum of eqs. (5.23), (5.26), (5.29).

6. A conjecture

A striking aspect of our result for higher derivative corrections is that it can be written in a form that only uses basic objects of 3-algebras: the covariant derivative on scalars and fermions, and the triple product $[X^I,X^J,X^K]$ and $[X^I,X^J,\lambda]$. To leading order in derivatives we have written the complete answer, for both bosons and fermions, and we expect it is maximally supersymmetric (though we did not prove that here).

Given this situation, it seems reasonable to speculate that the same derivative corrections are relevant to all 3-algebras with maximal supersymmetry, regardless of their signature. For Euclidean signature, this in fact only includes just one theory besides the ones we were considering, namely the Bagger-Lambert A_4 theory [3].[8] Thus we conjecture that the action in eqs. (3.14), (5.23), (5.26), (5.29) also embodies the derivative corrections to the Euclidean 3-algebra A_4 theory.

[7] More precisely we have $\epsilon^{\mu\nu\rho}\gamma_\rho = \gamma^{\mu\nu}$. Moreover one will drop all terms involving $\alpha'^2(\gamma_\mu\partial^\mu\lambda + g_{\rm YM}\gamma^i[X^i,\lambda])$.

[8] For arbitrary signature it is possible to construct more such algebras. In particular, algebras with $(2,p)$ signature have been classified in [28]. We would like to thank Jose Figueroa-O' Farrill for a comment on this point.

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It may legitimately be argued that there is no concrete test of this conjecture given that we do not presently know how to compute derivative corrections to the membrane field theory starting from M-theory. However an important test in our opinion will be whether the higher-derivative theory we have constructed is really maximally supersymmetric. Since our Lagrangian inherits its entire structure from $\mathcal{N} = 8$ SYM, this must surely be the case. Assuming supersymmetry can be proved, it is most likely that the proof will rely only on abstract 3-algebra properties and therefore will go through in the same way for the A_4 theory.

7. Conclusions

In this paper we have shown that the world-volume theory of multiple D2-branes in string theory, including both the $\mathcal{N} = 8$ SYM part as well as the leading (bosonic and fermionic) higher derivative corrections, is equivalent by a dNS duality to a derivative-corrected Lorentzian 3-algebra theory. This generalizes the result in [13] to incorporate α' corrections. We see no obstacle in principle to extending this to any finite order in α' as long as the D2-brane action is known to that order.

The result has the elegant feature that it depends only on 3-algebra quantities: the 3-bracket and covariant derivative. We have conjectured that it has more general significance than the context in which we have derived it. Extended supersymmetric CFT's in 3 dimensions appear to all depend on the 3-algebra structure (although if $\mathcal{N} < 8$ then some of the original 3-algebra assumptions need to be relaxed [19, 20]). Our results can be extended in a straightforward manner only to the Euclidean A_4 3-algebra but in the future, with extra work, it should be possible to extend them at least to the $\mathcal{N} = 6$ case.

Note added: while this article was being prepared ref. [29] appeared on the arXiv, in which a non-linear theory for multiple M2-branes has been proposed. Earlier papers that might be related are [30 – 32].

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References

[1] J. Bagger and N. Lambert, *Modeling multiple M2's*, *Phys. Rev.* **D 75** (2007) 045020 [hep-th/0611108].

[2] J. Bagger and N. Lambert, *Gauge symmetry and supersymmetry of multiple $M2$-branes*, *Phys. Rev.* **D 77** (2008) 065008 [arXiv:0711.0955].

[3] J. Bagger and N. Lambert, *Comments on multiple $M2$-branes*, *JHEP* **02** (2008) 105 [arXiv:0712.3738].

[4] A. Gustavsson, *Algebraic structures on parallel $M2$-branes*, arXiv:0709.1260.

[5] U. Gran, B.E.W. Nilsson and C. Petersson, *On relating multiple $M2$ and $D2$-branes*, arXiv:0804.1784.

[6] J. Gomis, G. Milanesi and J.G. Russo, *Bagger-Lambert theory for general Lie algebras*, *JHEP* **06** (2008) 075 [arXiv:0805.1012].

[7] S. Benvenuti, D. Rodriguez-Gomez, E. Tonni and H. Verlinde, $N = 8$ *superconformal gauge theories and $M2$ branes*, arXiv:0805.1087.

[8] P.-M. Ho, Y. Imamura and Y. Matsuo, *$M2$ to $D2$ revisited*, *JHEP* **07** (2008) 003 [arXiv:0805.1202].

[9] M.A. Bandres, A.E. Lipstein and J.H. Schwarz, *Ghost-free superconformal action for multiple $M2$-Branes*, *JHEP* **07** (2008) 117 [arXiv:0806.0054].

[10] J. Gomis, D. Rodriguez-Gomez, M. Van Raamsdonk and H. Verlinde, *Supersymmetric Yang-Mills theory from lorentzian three-algebras*, arXiv:0806.0738.

[11] S. Mukhi and C. Papageorgakis, *$M2$ to $D2$*, *JHEP* **05** (2008) 085 [arXiv:0803.3218].

[12] Y. Honma, S. Iso, Y. Sumitomo and S. Zhang, *Janus field theories from multiple $M2$ branes*, *Phys. Rev.* **D 78** (2008) 025027 [arXiv:0805.1895].

[13] B. Ezhuthachan, S. Mukhi and C. Papageorgakis, *$D2$ to $D2$*, *JHEP* **07** (2008) 041 [arXiv:0806.1639].

[14] E.A. Bergshoeff, M. de Roo, O. Hohm and D. Roest, *Multiple membranes from gauged supergravity*, *JHEP* **08** (2008) 091 [arXiv:0806.2584].

[15] H. Nicolai and H. Samtleben, *Chern-Simons vs. Yang-Mills gaugings in three dimensions*, *Nucl. Phys.* **B 668** (2003) 167 [hep-th/0303213].

[16] B. de Wit, I. Herger and H. Samtleben, *Gauged locally supersymmetric $D = 3$ nonlinear σ-models*, *Nucl. Phys.* **B 671** (2003) 175 [hep-th/0307006].

[17] B. de Wit, H. Nicolai and H. Samtleben, *Gauged supergravities in three dimensions: a panoramic overview*, hep-th/0403014.

[18] A.A. Tseytlin, *On non-abelian generalisation of the Born-Infeld action in string theory*, *Nucl. Phys.* **B 501** (1997) 41 [hep-th/9701125].

[19] M. Schnabl and Y. Tachikawa, *Classification of $N = 6$ superconformal theories of ABJM type*, arXiv:0807.1102.

[20] J. Bagger and N. Lambert, *Three-algebras and $N = 6$ Chern-Simons gauge theories*, arXiv:0807.0163.

[21] E.A. Bergshoeff, O. Hohm, D. Roest, H. Samtleben and E. Sezgin, *The superconformal gaugings in three dimensions*, arXiv:0807.2841.

[22] P.K. Townsend, *D-branes from M-branes*, *Phys. Lett.* **B 373** (1996) 68 [hep-th/9512062].

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[23] C. Schmidhuber, *D-brane actions*, *Nucl. Phys.* **B 467** (1996) 146 [hep-th/9601003].

[24] M. Aganagic, J. Park, C. Popescu and J.H. Schwarz, *Dual D-brane actions*, *Nucl. Phys.* **B 496** (1997) 215 [hep-th/9702133].

[25] M. Aganagic, C. Popescu and J.H. Schwarz, *Gauge-invariant and gauge-fixed D-brane actions*, *Nucl. Phys.* **B 495** (1997) 99 [hep-th/9612080].

[26] M. Cederwall, B.E.W. Nilsson and D. Tsimpis, $D = 10$ *super-Yang-Mills at* $O(\alpha^2)$, *JHEP* **07** (2001) 042 [hep-th/0104236].

[27] E.A. Bergshoeff, A. Bilal, M. de Roo and A. Sevrin, *Supersymmetric non-abelian Born-Infeld revisited*, *JHEP* **07** (2001) 029 [hep-th/0105274].

[28] P. de Medeiros, J.M. Figueroa-O'Farrill and E. Mendez-Escobar, *Metric Lie* 3*-algebras in Bagger-Lambert theory*, arXiv:0806.3242.

[29] R. Iengo and J.G. Russo, *Non-linear theory for multiple* $M2$ *branes*, arXiv:0808.2473.

[30] T. Li, Y. Liu and D. Xie, *Multiple* $D2$*-brane action from* $M2$*-branes*, arXiv:0807.1183.

[31] J. Kluson, $D2$ *to* $M2$ *procedure for* $D2$*-brane DBI effective action*, arXiv:0807.4054.

[32] G. Bonelli, A. Tanzini and M. Zabzine, *Topological branes, p-algebras and generalized Nahm equations*, arXiv:0807.5113.